M32

Coimisiún na Scrúduithe Stáit
State Examinations Commission

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Marking Scheme	*Leaving Certificate Examination, 2003*
Applied Mathematics	*Higher Level*

General Guidelines

1 Penalties of three types are applied to candidates' work as follows:

Slips	- numerical slips	S(-1)
Blunders	- mathematical errors	B(-3)
Misreading	- if not serious	M(-1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (att 2).

2 Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

3 Mark scripts in red unless candidate uses red. If a candidate uses red, mark the script in blue or black.

4 Number the grid on each script 1 to 10 in numerical order, not the order of answering.

5 Scrutinise **all** pages of the answer book.

6 The marking scheme shows one correct solution to each question. In many cases there are other equally valid methods.

1. **(a)** The points p, q and r all lie in a straight line.
A train passes point p with speed u m/s. The train is travelling with uniform retardation f m/s^2. The train takes 10 seconds to travel from p to q and 15 seconds to travel from q to r, where $|pq| = |qr| = 125$ metres.

(i) Show that $f = \dfrac{1}{3}$.

(ii) The train comes to rest s metres after passing r.
Find s, giving your answer correct to the nearest metre.

(i)

$$s = ut + \tfrac{1}{2}at^2$$

$$125 = u(10) - \tfrac{1}{2}(f)10^2 \qquad 5$$
$$250 = u(25) - \tfrac{1}{2}(f)25^2 \qquad 5$$

$$12.5 = u - 5f$$
$$10 = u - 12.5f$$

$$f = \frac{1}{3} \qquad 5$$

$$u = \frac{85}{6} \quad \text{or} \quad 14.17$$

(ii)

$$v^2 = u^2 + 2as$$

$$0 = \left(\frac{85}{6}\right)^2 + 2\left(-\frac{1}{3}\right)(250 + s)$$

$$\Rightarrow \qquad 250 + s = 301.04 \text{ m}$$

$$\Rightarrow \qquad s = 51 \text{ m} \qquad 5 \qquad \boxed{20}$$

(b) A man runs at constant speed to catch a bus.
At the instant the man is 40 metres from the bus, it begins to accelerate uniformly from rest away from him.
The man just catches the bus 20 seconds later.

(i) Find the constant speed of the man.

(ii) If the constant speed of the man had instead been 3 m/s, show that the closest he gets to the bus is 17.5 metres.

(i) Bus	$v = u + at$	$\Rightarrow$	$v = 0 + 20a$		
	$s = ut + \frac{1}{2}at^2$	$\Rightarrow$	$s_{bus} = 0 + \frac{1}{2}a(20)^2$		
Man	$v = u + at$	$\Rightarrow$	$v = u + 0$		
	$s = ut + \frac{1}{2}at^2$	$\Rightarrow$	$s_{man} = 20u + 0$		
After 20 seconds	$s_{man} = 40 + s_{bus}$	$\Rightarrow$	$20u = 40 + 200a$	5	
	$v_{man} = v_{bus}$	$\Rightarrow$	$u = 20a$	5	
		$\Rightarrow$	$u = 4\ m/s\ \ (and\ \ a = 0.2)$	5	
(ii)	distance		$= 40 + s_{bus} - \mathrm{s}_{\mathrm{man}}$		
			$= 40 + \left(0 + \frac{1}{2}at^2\right) - (ut + 0)$		
			$= 40 + 0.1t^2 - 3t$	5	
	$\frac{d(\text{distance})}{dt}$		$= 0.2t - 3 = 0$ when $t = 15$	5	
	closest distance		$= 40 + (0.1)(15)^2 - 3(15)$		
			$= 17.5\ \mathrm{m}$	5	30

2. **(a)** A woman travelling north at 10 km/h finds that the wind appears to blow from the west. When the woman trebles her speed, the wind appears to blow from the north-west.

Find the velocity of the wind.

$$\vec{V}_P = 10\ \vec{j}$$
$$\vec{V}_{WP} = x\ \vec{i}$$ 5
$$\vec{V}_W = a\ \vec{i} + b\ \vec{j}$$

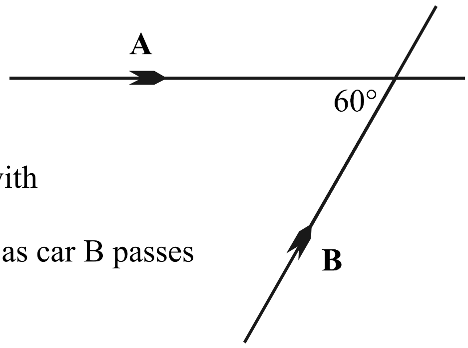
$$\vec{V}_{WP} = \vec{V}_W - \vec{V}_P$$
$$x\ \vec{i} = a\ \vec{i} + (b-10)\ \vec{j}$$ 5
$$\Rightarrow \quad b = 10.$$

$$\vec{V}_P = 30\ \vec{j}$$
$$\vec{V}_{WP} = y\ \vec{i} - y\ \vec{j}$$
$$\vec{V}_W = a\ \vec{i} + b\ \vec{j}$$

$$\vec{V}_{WP} = \vec{V}_W - \vec{V}_P$$
$$y\ \vec{i} - y\ \vec{j} = a\ \vec{i} + (b-30)\ \vec{j}$$ 5
$$\Rightarrow \quad y = 30 - b = 20$$
$$\Rightarrow \quad a = y = 20$$ 5

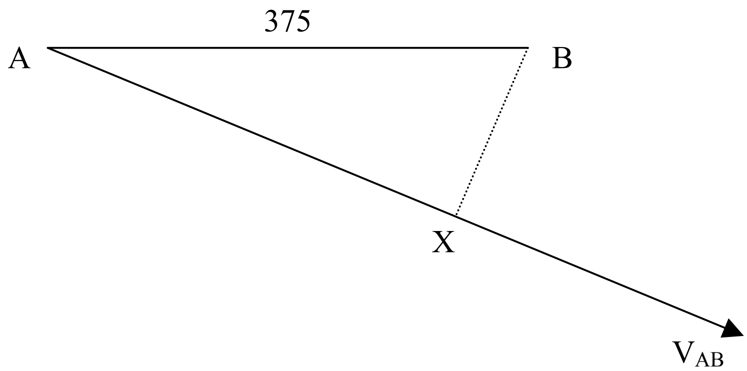
$$\Rightarrow \quad \vec{V}_W = 20\ \vec{i} + 10\ \vec{j} \quad \text{km/h}$$ 20

(b) Two straight roads intersect at an angle of 60°. Car A is moving towards the intersection with a uniform speed of 7.5 m/s.
Car B is moving towards the intersection with a uniform speed of 10 m/s.
Car A is 375 m away from the intersection as car B passes the intersection.



(i) Find the velocity of car A relative to car B.

(ii) How far is each car from the intersection at the instant when they are nearest to each other?
Give your answers correct to the nearest metre.



(i)

$$\vec{V}_A = 7.5\ \vec{i}$$
$$\vec{V}_B = 10\cos 60\ \vec{i} + 10\sin 60\ \vec{j} \qquad 5$$

$$\vec{V}_{AB} = \vec{V}_A - \vec{V}_B$$
$$= (7.5 - 10\cos 60)\ \vec{i} - 10\sin 60\ \vec{j} \qquad 5$$
$$= 2.5\ \vec{i} - 5\sqrt{3}\ \vec{j}$$

$$\left.\begin{aligned} \text{magnitude} &= \sqrt{(2.5)^2 + \left(5\sqrt{3}\right)^2} = 9.01 \\ \text{direction} &= \tan^{-1}\left(2\sqrt{3}\right) \text{ or } 73.9° \text{ S. of E.} \end{aligned}\right\} \qquad 5$$

(ii)

$$|\text{AX}| = 375\cos 73.9° \text{ or } 103.99 \qquad 5$$

$$\text{time} = \frac{|\text{AX}|}{\text{V}_{\text{AB}}} = \frac{103.99}{9.01} = 11.54 \qquad 5$$

$$\left.\begin{aligned} \text{distance of A from intersection} &= 375 - 7.5(11.54) = 288 \text{ m} \\ \text{distance of B from intersection} &= 10(11.54) = 115 \text{ m} \end{aligned}\right\} \qquad 5$$

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3. **(a)** A particle is projected from a point on level horizontal ground at an angle θ to the horizontal ground.
Find θ, if the horizontal range of the particle is five times the maximum height reached by the particle.

$$r = \{u\cos\theta(t)\}\vec{i} + \{u\sin\theta(t) - \tfrac{1}{2}gt^2\}\vec{j}$$

Range $r_{\vec{j}} = 0$

$$\Rightarrow \quad t = \frac{2u\sin\theta}{g} \qquad 5$$

$$\text{Range} = \{u\cos\theta\}\left\{\frac{2u\sin\theta}{g}\right\} \qquad 5$$

$$= \frac{2u^2\sin\theta\cos\theta}{g}$$

Max height $t = \dfrac{u\sin\theta}{g}$ 5

$$\text{Max height} = \{u\sin\theta\}\left\{\frac{u\sin\theta}{g}\right\} - \tfrac{1}{2}g\left\{\frac{u\sin\theta}{g}\right\}^2 \qquad 5$$

$$= \frac{u^2\sin^2\theta}{g} - \frac{u^2\sin^2\theta}{2g}$$

$$= \frac{u^2\sin^2\theta}{2g}$$

$$\text{Range} = 5 \times \text{Maximum height}$$

$$\frac{2u^2\sin\theta\cos\theta}{g} = \frac{5u^2\sin^2\theta}{2g}$$

$$\Rightarrow \quad \tan\theta = \tfrac{4}{5} \quad \text{or} \quad \theta = 38.7^\circ \qquad 5 \qquad \boxed{25}$$

(b) A particle is projected up an inclined plane with initial velocity u m/s. The line of projection makes an angle α **with the horizontal** and the inclined plane makes an angle β with the horizontal. (The plane of projection is vertical and contains the line of greatest slope.)

Find, in terms of u, g, α and β, the range of the particle up the inclined plane.

$$r_j = 0$$ 5

$$\text{usin}(\alpha - \beta).\text{t} - \tfrac{1}{2} g \cos\beta.\text{t}^2 = 0$$ 5

$$\Rightarrow \quad \text{t} = \frac{2\text{usin}(\alpha - \beta)}{\text{gcos}\beta}$$ 5

Range $= r_i$

$$= \text{ucos}(\alpha - \beta).\text{t} - \tfrac{1}{2} g \sin\beta.\text{t}^2$$ 5

$$= \text{ucos}(\alpha - \beta).\left\{\frac{2\text{usin}(\alpha - \beta)}{\text{gcos}\beta}\right\} - \tfrac{1}{2} g \sin\beta..\left\{\frac{2\text{usin}(\alpha - \beta)}{\text{gcos}\beta}\right\}^2$$ 5

$$= \frac{\text{u}^2}{\text{gcos}^2\beta}\{2\sin(\alpha - \beta)\}\{\cos\beta\cos(\alpha - \beta) - \sin\beta\sin(\alpha - \beta)\}$$

$$= \frac{\text{u}^2}{\text{gcos}^2\beta}\{2\sin(\alpha - \beta)\}\{\cos(2\beta - \alpha)\}$$

$$= \frac{\text{u}^2}{\text{gcos}^2\beta}\{\sin(2\alpha - \beta) - \sin\beta\}$$

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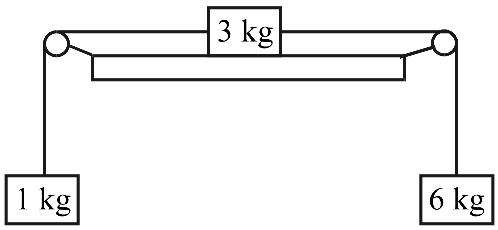
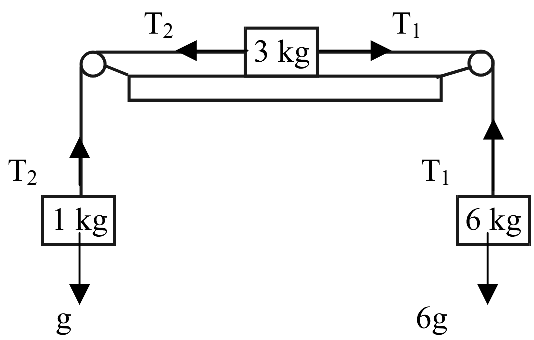
4. **(a)** A particle of mass 3 kg rests on a smooth horizontal table and is attached by two light inelastic strings to particles of masses 6 kg and 1 kg which hang over smooth light pulleys at opposite edges of the table.
The system is released from rest.



Find the acceleration of the system, in terms of *g*.



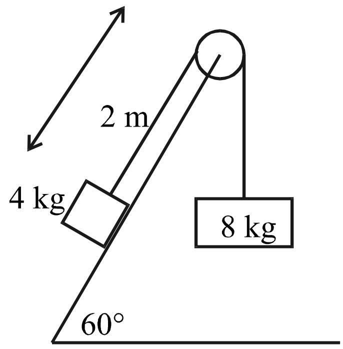
$$6g - T_1 = 6a$$ 5

$$T_1 - T_2 = 3a$$ 5

$$T_2 - g = a$$ 5

$$5g = 10a$$
$$a = \tfrac{1}{2}g \ \ ms^{-2}$$ 5 | 20

(b) A block of mass 4 kg rests on a rough plane inclined at 60° to the horizontal. It is connected by a light inextensible string which passes over a smooth, light, fixed pulley to a particle of mass 8 kg which hangs freely under gravity.
The coefficient of friction between the block and the plane is $\frac{1}{4}$. The system starts from rest with the block at a distance of 2 m from the pulley.
The 8 kg mass moves vertically downwards.



(i) Show that the tension in the string is 52 N, correct to the nearest whole number.
(ii) How far has the block moved up the plane after 1 second?
(iii) After 1 second the string is cut. Determine whether or not the block will reach the pulley.

(i) $8g - T = 8a$ 5

$T - 4g\sin 60 - \frac{1}{4}(4g\cos 60) = 4a$ 5

$T = 52$ N 5

$a = 3.3\, ms^{-2}$

(ii) First second $s = ut + \frac{1}{2}at^2$
$= 0 + \frac{1}{2}(3.3)(1)$
$= 1.65$ m 5

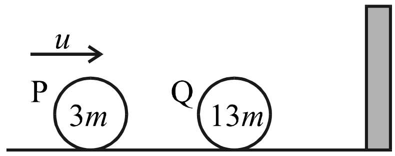
$v = u + at$
$= 0 + 3.3(1)$
$= 3.3\, ms^{-1}$

(iii) Second stage $v^2 = u^2 + 2as$
$= 3.3^2 + 2(-g\sin 60 - \frac{1}{8}g)(0.35)$ 5
$= 4.1\, ms^{-1}$

As $v^2 > 0$
$\Rightarrow$ block will reach the pulley 5

30

5. **(a)** A smooth sphere P, of mass $3m$, moving with speed u, collides directly with a smooth sphere Q, of mass $13m$, which is at rest. Sphere Q then collides with a vertical wall which is perpendicular to the direction of motion of the spheres. The coefficient of restitution for all of the collisions is e.



Find

(i) the speed, in terms of u and e, of each sphere after the first collision

(ii) the range of values of e for which there will be a second collision between the spheres.

(i) PCM $\quad 3m(u)+13m(0) = 3mv_1 + 13mv_2$ 5

NEL $\quad v_1 - v_2 = -e(u-0)$ 5

$$\left.\begin{aligned} v_1 &= \frac{u}{16}(3-13e) \\ v_2 &= \frac{3u}{16}(1+e) \end{aligned}\right\}$$ 5

(ii) There will be a second collision if :

$$ev_2 > -v_1$$ 5

$$e\frac{3u}{16}(1+e) > -\frac{u}{16}(3-13e)$$

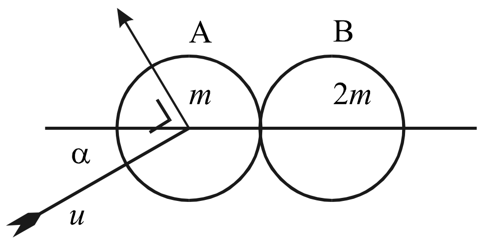
$$3e^2 - 10e + 3 > 0$$

$$(3e-1)(e-3) > 0$$

$$\Rightarrow \quad 0 \le e < \tfrac{1}{3}$$ 5

25

(b) A smooth sphere A, of mass m, moving with speed u, collides with a smooth sphere B, of mass $2m$, which is at rest. The direction of motion of A, before impact, makes an angle α with the line of centres at impact, where $0° \leq \alpha < 90°$. As a result of the collision, the direction of A is deflected through an angle of 90°. The coefficient of restitution between the spheres is e.



(i) Show that $\tan\alpha = \sqrt{\dfrac{2e-1}{3}}$.

(ii) Find e, if the magnitude of the impulse exerted by A on B is $mu\cos\alpha$.

(i) PCM $\quad mu\cos\alpha + 2m(0) = mv_1 + 2mv_2$ — 5

NEL $\quad v_1 - v_2 = -e\,(u\cos\alpha - 0)$ — 5

$$\left.\begin{aligned} v_1 &= \frac{u\cos\alpha\{1-2e\}}{3} \\ v_2 &= \frac{u\cos\alpha\{1+e\}}{3} \end{aligned}\right\}$$ — 5

$$\left.\begin{aligned} \tan\alpha &= \frac{-v_1}{u\sin\alpha} \\ &= \frac{-u\cos\alpha\{1-2e\}}{3u\sin\alpha} \\ \tan^2\alpha &= \frac{2e-1}{3} \\ \tan\alpha &= \sqrt{\frac{2e-1}{3}} \end{aligned}\right\}$$ — 5

(ii) Impulse exerted by A on B $= |2m(0) - 2mv_2|$

$$mu\cos\alpha = \frac{2mu\cos\alpha\{1+e\}}{3}$$

$$3 = 2 + 2e$$

$$\Rightarrow \quad e = \tfrac{1}{2}$$ — 5

25

6. **(a)** A particle is moving with simple harmonic motion of period π seconds about a fixed point o.
The maximum speed of the particle is 8 cm/s.
(i) Find the amplitude of the motion.
(ii) Find the speed of the particle when it is at a distance of 3 cm from o.

$$\text{Period} = \frac{2\pi}{\omega}$$

$$\pi = \frac{2\pi}{\omega}$$

$$\Rightarrow \quad \omega = 2 \qquad 5$$

$$v_{\max} = \omega a$$

$$8 = 2a$$

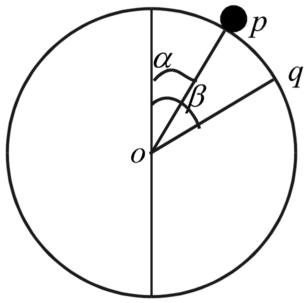
$$\Rightarrow \quad a = 4 \text{ cm} \qquad 5$$

$$v = \omega\sqrt{a^2 - x^2} \qquad 5$$

$$= 2\sqrt{16-9}$$

$$\Rightarrow \quad v = 2\sqrt{7} \text{ cm/s} \qquad 5 \qquad 20$$

(b) A particle of mass m is held at a point p on the surface of a fixed smooth sphere, centre o and radius r.
op makes an angle α with the upward vertical.
The particle is released from rest.
When the particle reaches an arbitrary point q, its speed is v.
oq makes an angle β with the upward vertical.



(i) Show that $v^2 = 2gr(\cos\alpha - \cos\beta)$.

(ii) If $\cos\alpha = \dfrac{2}{3}$ and if q is the point at which the particle leaves the surface, find the value of β.

(i)

$$\left.\begin{aligned} \text{KE at } p &= 0 \\ \text{KE at } q &= \tfrac{1}{2}mv^2 \end{aligned}\right\} \quad 5$$

$$\left.\begin{aligned} \text{PE at } p &= mgr\cos\alpha \\ \text{PE at } q &= mgr\cos\beta \end{aligned}\right\} \quad 5$$

$$\begin{aligned} \text{Energy at } q &= \text{Energy at } p \\ \tfrac{1}{2}mv^2 + mgr\cos\beta &= mgr\cos\alpha \\ v^2 &= 2gr\{\cos\alpha - \cos\beta\} \end{aligned} \quad 5$$

(ii)

$$mg\cos\beta - R = \frac{mv^2}{r} \quad 5$$

$$R = 0 \quad 5$$

$$\begin{aligned} mg\cos\beta - 0 &= \frac{m2gr\{\cos\alpha - \cos\beta\}}{r} \\ \cos\beta &= 2\{\tfrac{2}{3} - \cos\beta\} \\ 3\cos\beta &= \tfrac{4}{3} \\ &\Rightarrow \beta = \cos^{-1}\left(\tfrac{4}{9}\right) \text{ or } 63.6° \end{aligned} \quad 5$$

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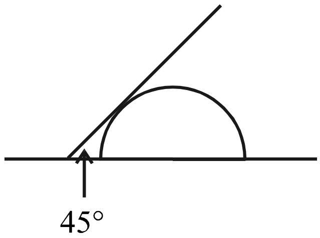
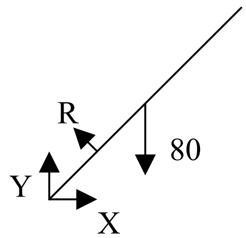
7. **(a)** A uniform rod of length 3 m and weight 80 N is smoothly hinged at one end to a rough horizontal floor. The rod rests on the smooth curved surface of a hemisphere whose plane face is on the floor. The coefficient of friction between the plane surface of the sphere and the floor is μ.
The rod and the centre of the hemisphere lie in the same vertical plane.
The rod is in equilibrium inclined at 45° to the horizontal. The hemisphere, of radius 1 m and weight 40 N, is in limiting equilibrium (that is, just on the point of slipping).

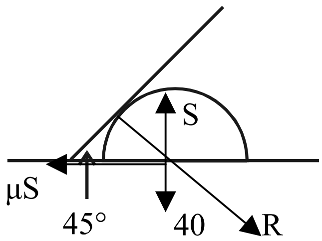


(i) Show, on separate diagrams, all the forces acting on the rod and the hemisphere.

(ii) Show that the reaction between the rod and the hemisphere is $60\sqrt{2}$ N.

(iii) Find the coefficient of friction, μ, between the hemisphere and the floor.





5, 5

(ii) Take moments about a for rod:

$$R\,(1) = 80 \times (1.5\cos 45)$$

$$\Rightarrow \quad R = 60\sqrt{2} \text{ N}$$ 5

(iii) Resolve forces acting on hemisphere:

vert: $S = 40 + R\cos 45$

$$= 40 + 60\sqrt{2}\left\{\frac{1}{\sqrt{2}}\right\}$$

$$= 100 \text{ N}$$ 5

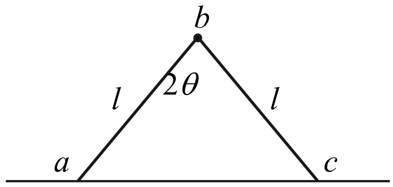
horiz: $\mu S = R\sin 45$

$$\mu(100) = 60\sqrt{2}\left\{\frac{1}{\sqrt{2}}\right\}$$

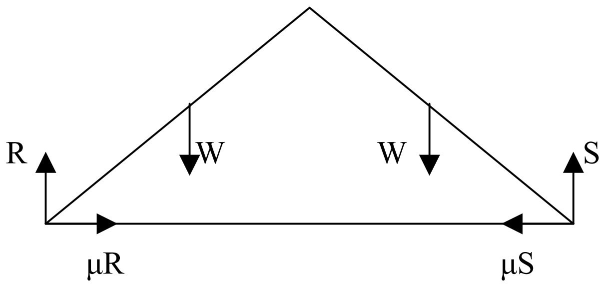
$$\Rightarrow \quad \mu = \tfrac{3}{5}$$ 5

25

(b) Two uniform ladders, [ab] and [bc], each of weight W and length l, are smoothly jointed at b. They rest in a vertical plane with a and c on rough horizontal ground.
The coefficient of friction at both a and c is μ.
Let $|\angle abc| = 2\theta$.



Show that $\mu \geq \frac{1}{2}\tan\theta$.



Take moments about a for system:

$$W(1) + W(3) = S(4)$$

$$\Rightarrow \quad S = W$$ 5

Resolve forces acting on system:

vert: $\quad R + S = W + W$

$$\Rightarrow \quad R = W$$ 5

Take moments about b for rod ba:

$$R(\ell\sin\theta) = W\left(\tfrac{1}{2}\ell\sin\theta\right) + \mu R(\ell\cos\theta)$$ 5,5

$$W\tan\theta = \tfrac{1}{2}W\tan\theta + \mu W$$

$$\mu = \tfrac{1}{2}\tan\theta$$ 5

$$\Rightarrow \quad \mu \geq \tfrac{1}{2}\tan\theta$$

25

8. **(a)** Prove that the moment of inertia of a uniform rod of mass m and length $2l$ about an axis through its centre perpendicular to the rod is $\frac{1}{3}ml^2$.

Let M = mass per unit length

mass of element $= M\{dx\}$

moment of inertia of the element $= M\{dx\}x^2$ 5

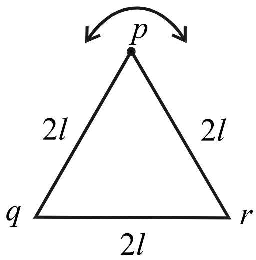
moment of inertia of the rod $= M \int_{-\ell}^{\ell} x^2\, dx$ 5

$= M\left[\frac{x^3}{3}\right]_{-\ell}^{\ell}$ 5

$= \frac{2}{3} M\ell^3$

$= \frac{1}{3} m\ell^2$ 5 | 20

(b) Three rods, each of mass m and length $2l$, are jointed together at their ends to form a triangle pqr. The triangle is free to rotate about a fixed horizontal axis through p perpendicular to its plane.



(i) Find, in terms of l, the period of small oscillations.

(ii) Find, in terms of l, the length of the equivalent simple pendulum.

(i)

$$\left.\begin{aligned} I_{pq} &= \tfrac{4}{3}m\ell^2 \\ I_{pr} &= \tfrac{4}{3}m\ell^2 \\ I_{qr} &= \tfrac{1}{3}m\ell^2 + m\left\{\sqrt{3}\,\ell\right\}^2 \end{aligned}\right\}$$ 5

$$I_p = I_{pq} + I_{pr} + I_{qr}$$
$$= 6m\ell^2$$

$$Mgh = \frac{mg\sqrt{3}\,\ell}{2} + \frac{mg\sqrt{3}\,\ell}{2} + mg\sqrt{3}\,\ell$$ 5
$$= 2mg\sqrt{3}\,\ell$$

$$T = 2\pi\sqrt{\frac{I}{Mgh}}$$ 5

$$= 2\pi\sqrt{\frac{6m\ell^2}{2mg\sqrt{3}\,\ell}} \quad \text{or} \quad 2\pi\sqrt{\frac{\sqrt{3}\,\ell}{g}}$$ 5

(ii)

$$2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{\sqrt{3}\,\ell}{g}}$$ 5

$$L = \sqrt{3}\,\ell$$ 5

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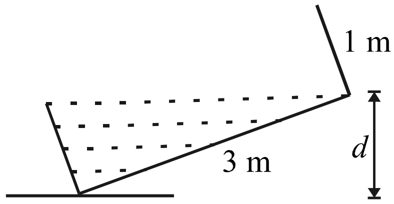
9. **(a)** A rectangular tank is 1 m high and has a base 2 m by 3 m. It is half filled with water and is tilted about one of the shorter edges of its base until the water just begins to overflow. When the tank is tilted to this position, a diagonal of its vertical faces lies along the surface of the water.
With the tank held in this position,



(i) show that the depth, d, of the water in the tank is $\dfrac{3}{\sqrt{10}}$ m

(ii) show that the thrust on the base of the tank is $900\sqrt{10}g$ N

(iii) find the thrust on a vertical side of the tank (e.g., the side facing towards you in the diagram).

(i) area of triangle $= \frac{1}{2}(1)(3)$

$$\tfrac{1}{2}(d)\left(\sqrt{10}\right) = \tfrac{1}{2}(1)(3)$$

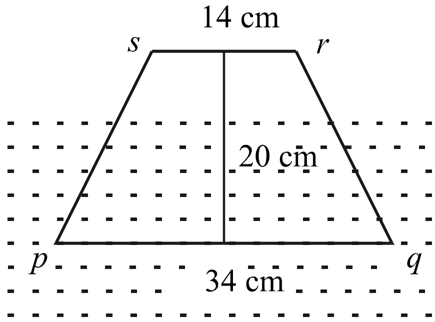
$$d = \frac{3}{\sqrt{10}}$$ 5

(ii)
$$\text{Thrust} = \text{Pressure} \times \text{Area}$$
$$= \rho g\left(\tfrac{1}{2}d\right) \times \{3 \times 2\}$$ 5
$$= 1000\,g\left(\frac{3}{2\sqrt{10}}\right) \times 6$$
$$= 900\sqrt{10}\,g$$ 5

(iii)
$$\text{Thrust} = \text{Pressure} \times \text{Area}$$
$$= \rho g\left(\tfrac{1}{3}d\right) \times \left\{\tfrac{1}{2}(1)(3)\right\}$$ 5
$$= 1000\,g\left(\frac{3}{3\sqrt{10}}\right) \times \frac{3}{2}$$
$$= 150\sqrt{10}\,g$$ 5

25

(b) The trapezium *pqrs* is a cross-section of a block of wood. The distance between the parallel sides [*pq*] and [*rs*] is 20 cm. $|pq| = 34$ cm, $|rs| = 14$ cm and $|ps| = |qr|$. The relative density of the wood is $\frac{7}{10}$. The block floats with the face containing [*pq*] immersed in water and horizontal.



Find the depth of side [*pq*] below the surface of the water.

Let depth of $[pq]$ below the surface $= h$

$$\text{Buoyancy } B = \text{weight of liquid displaced}$$
$$= \rho V g$$
$$= 1000\left\{\tfrac{1}{2}(34 + 34 - h)h\right\}\{x\}g \qquad 5$$
$$= 1000\left\{34h - \tfrac{1}{2}h^2\right\}\{x\}g$$

$$\text{Weight of block of wood} \quad W = \rho V g$$
$$= 700\left\{\tfrac{1}{2}(34 + 14)20\right\}\{x\}g \qquad 5$$
$$= 700\{480\}\{x\}g$$

$$B = W \qquad 5$$
$$1000\left\{34h - \tfrac{1}{2}h^2\right\}\{x\}g = 700\{480\}\{x\}g$$

$$34h - \tfrac{1}{2}h^2 = 336 \qquad 5$$

$$h = 12\text{ cm} \qquad 5$$

25

10. **(a)** Solve the differential equation

$$\frac{dy}{dx} = \frac{xy}{2x^2 - 3}$$

given that $y = 1$ when $x = \sqrt{2}$.

$$\frac{dy}{dx} = \frac{x\,y}{2x^2 - 3}$$

$$\int \frac{dy}{y} = \int \frac{x\,dx}{2x^2 - 3} \qquad 5$$

$$\ln \mathrm{y} = \tfrac{1}{4}\ln\left(2x^2 - 3\right) + \mathrm{C} \qquad 5$$

$$y = 1, x = \sqrt{2} \Rightarrow \mathrm{C} = 0 \qquad 5$$

$$\ln \mathrm{y} = \tfrac{1}{4}\ln\left(2x^2 - 3\right)$$

$$\mathrm{y} = \left(2x^2 - 3\right)^{\frac{1}{4}} \qquad 5 \qquad \boxed{20}$$

(b) A car of mass 490 kg moves along a straight level horizontal road against a resistance of $70v$ N, where v m/s is the speed of the car.
The engine exerts a constant power of 63 kW.

(i) Show that the equation of motion is

$$7\frac{dv}{dt} = \frac{900 - v^2}{v}.$$

(ii) Calculate, correct to two decimal places, the time it takes the car to increase its speed from 10 m/s to 20 m/s.

(i)

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

$$\frac{63000}{v} - 70v = 490\frac{dv}{dt} \qquad 5$$

$$\frac{900}{v} - v = 7\frac{dv}{dt}$$

$$7\frac{dv}{dt} = \frac{900 - v^2}{v} \qquad 5$$

(ii)

$$7\frac{dv}{dt} = \frac{900 - v^2}{v}$$

$$\int \frac{7v\,dv}{900 - v^2} = \int dt \qquad 5$$

$$\left[-\tfrac{7}{2}\ell n(900 - v^2)\right]_{10}^{20} = t \qquad 5$$

$$-\tfrac{7}{2}\ell n(900 - 400) + \tfrac{7}{2}\ell n(900 - 100) = t \qquad 5$$

$$-\tfrac{7}{2}\ell n(500) + \tfrac{7}{2}\ell n(800) = t$$

$$\tfrac{7}{2}\ell n\left(\frac{800}{500}\right) = t$$

$$t = 1.65 \qquad 5$$

30

Some Alternative Solutions

1. **(a)** The points p, q and r all lie in a straight line.
A train passes point p with speed u m/s. The train is travelling with uniform retardation f m/s^2. The train takes 10 seconds to travel from p to q and 15 seconds to travel from q to r, where $|pq| = |qr| = 125$ metres.

(i) Show that $f = \dfrac{1}{3}$.

(ii) The train comes to rest s metres after passing r.
Find s, giving your answer correct to the nearest metre.

(ii)

$$v = u + at$$

$$0 = \left(\frac{85}{6}\right) + \left(-\frac{1}{3}\right)(t)$$

$$\Rightarrow \qquad t = \frac{85}{2}$$

$$250 + s = \left(\frac{85}{6}\right)\left(\frac{85}{2}\right) + \frac{1}{2}\left(-\frac{1}{3}\right)\left(\frac{85}{2}\right)^2$$

$$= 301.04$$

$$\Rightarrow \qquad s = 51\,\text{m}$$

5

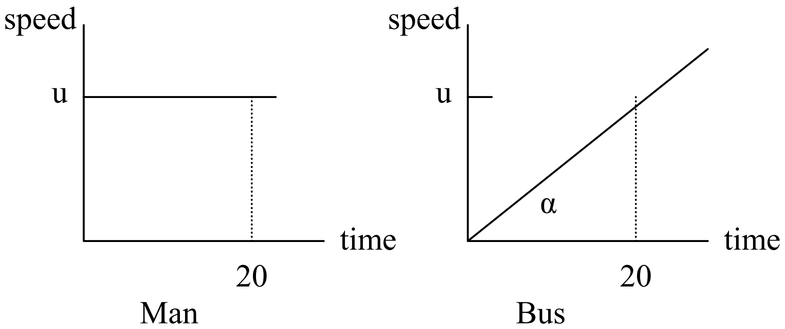
1.

(b) A man runs at constant speed to catch a bus.
At the instant the man is 40 metres from the bus, it begins to accelerate uniformly from rest away from him.
The man just catches the bus 20 seconds later.

(i) Find the constant speed of the man.

(ii) If the constant speed of the man had instead been 3 m/s, show that the closest he gets to the bus is 17.5 metres.

1. (b) (i)



5

After 20 seconds $\quad s_{man} = 40 + s_{bus}$ 5

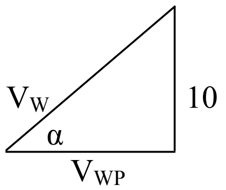
$\Rightarrow \quad 20u = 40 + \frac{1}{2}(20)u$

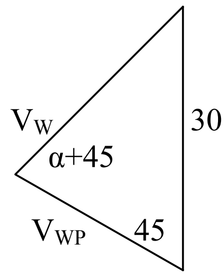
$\Rightarrow \quad u = 4\ m/s$ 5

$\left(and\ \ a\ =\ \tan\alpha\ =\ \frac{4}{20} = \frac{1}{5} \right)$ 5

2. **(a)** A woman travelling north at 10 km/h finds that the wind appears to blow from the west. When the woman trebles her speed, the wind appears to blow from the north-west.

Find the velocity of the wind.





$$\frac{\vec{V}_W}{\sin 90} = \frac{10}{\sin\alpha}$$ 5

$$\frac{\vec{V}_W}{\sin 45} = \frac{30}{\sin(\alpha+45)}$$ 5

$$\Rightarrow \quad \frac{10}{\sin\alpha} = \frac{30}{\sin(\alpha+45)}$$

$$10(\sin\alpha\cos 45 + \cos\alpha\sin 45) = 30\sin\alpha$$
$$10(\sin\alpha + \cos\alpha) = 30\sin\alpha$$

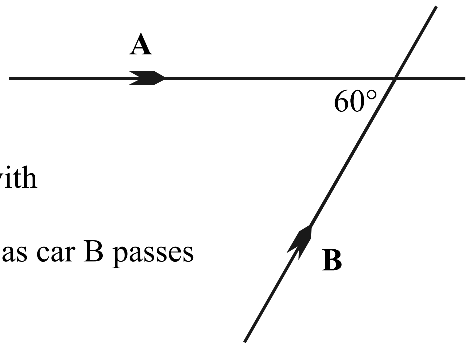
$$\Rightarrow \quad \tan\alpha = \tfrac{1}{2}$$ 5

$$\Rightarrow \quad \left|\vec{V}_W\right| = 10\sqrt{5} \quad \text{km/h}$$ 5 | 20

.

2.

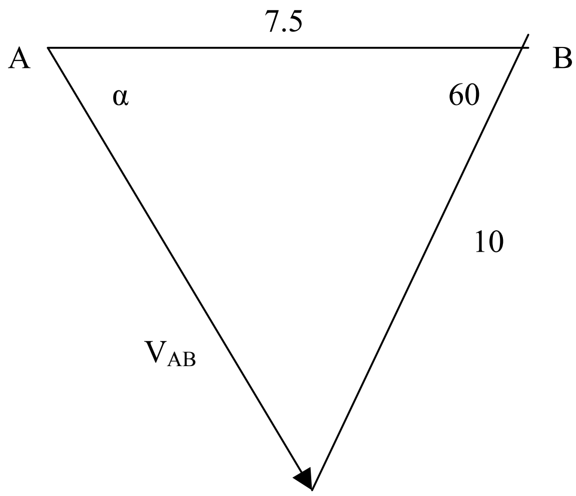
(b) Two straight roads intersect at an angle of 60°. Car A is moving towards the intersection with a uniform speed of 7.5 m/s.
Car B is moving towards the intersection with a uniform speed of 10 m/s.
Car A is 375 m away from the intersection as car B passes the intersection.



(i) Find the velocity of car A relative to car B.

(ii) How far is each car from the intersection at the instant when they are nearest to each other?
Give your answers correct to the nearest metre.

(i)



$$\left|\vec{V}_{AB}\right| = \sqrt{7.5^2 + 10^2 - 2(7.5)(10)\cos 60°}$$ 5

$$\frac{\sin\alpha}{10} = \frac{\sin 60}{\left|\vec{V}_{AB}\right|}$$ 5

$$\left|\vec{V}_{AB}\right| = 9.01$$

$$\text{direction} = \sin^{-1}\left(\frac{5\sqrt{3}}{9.01}\right) \text{ or } 73.9° \text{ S. of E.}$$ 5

3. **(a)** A particle is projected from a point on level horizontal ground at an angle θ to the horizontal ground.
Find θ, if the horizontal range of the particle is five times the maximum height reached by the particle.

$$r = \{u\cos\theta(t)\}\,\vec{i} + \{u\sin\theta(t) - \tfrac{1}{2}gt^2\}\,\vec{j}$$

Range $\quad r_{\vec{j}} = 0$

$$\Rightarrow \quad t = \frac{2u\sin\theta}{g} \qquad 5$$

$$\Rightarrow \quad \text{Range} = \{u\cos\theta\}\{\text{t}\} \qquad 5$$

$$\text{Max height} = \{u\sin\theta\}\{\text{t}_1\} - \tfrac{1}{2}g\{\text{t}_1\}^2 \qquad 5$$

$$\text{t}_1 = \frac{\text{t}}{2} \qquad 5$$

$$\text{Range} = 5 \times \text{Maximum height}$$

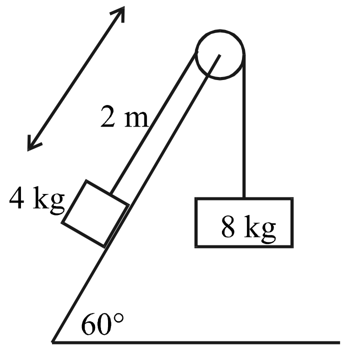
$$\{u\cos\theta\}\{\text{t}\} = 5 \times \left[\{u\sin\theta\}\left\{\frac{t}{2}\right\} - \tfrac{1}{2}g\left\{\frac{t}{2}\right\}^2\right]$$

$$2\{u\cos\theta\} = 5 \times \{u\sin\theta\} - \tfrac{5}{4}g\,t$$

$$2\{u\cos\theta\} = 5 \times \{u\sin\theta\} - \tfrac{5}{4}g\left(\frac{2u\sin\theta}{g}\right)$$

$$\Rightarrow \quad \tan\theta = \tfrac{4}{5} \quad \text{or} \quad \theta = 38.7^\circ \qquad 5 \qquad \boxed{25}$$

4.(b) A block of mass 4 kg rests on a rough plane inclined at 60° to the horizontal. It is connected by a light inextensible string which passes over a smooth, light, fixed pulley to a particle of mass 8 kg which hangs freely under gravity.
The coefficient of friction between the block and the plane is $\frac{1}{4}$. The system starts from rest with the block at a distance of 2 m from the pulley.
The 8 kg mass moves vertically downwards.



(i) Show that the tension in the string is 52 N, correct to the nearest whole number.
(ii) How far has the block moved up the plane after 1 second?
(iii) After 1 second the string is cut. Determine whether or not the block will reach the pulley.

(iii) Second stage

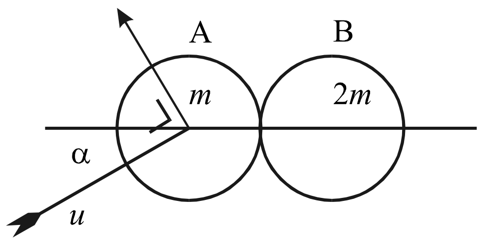
$$v^2 = u^2 + 2as$$
$$0 = 3.3^2 + 2\left(-g\sin 60 - \tfrac{1}{8}g\right)(s)$$
$$s = 0.56$$ 5

As $1.65 + 0.56 > 2$

$\Rightarrow$ block will reach the pulley 5

5.

(b) A smooth sphere A, of mass m, moving with speed u, collides with a smooth sphere B, of mass $2m$, which is at rest.
The direction of motion of A, before impact, makes an angle α with the line of centres at impact, where $0° \le \alpha < 90°$.
As a result of the collision, the direction of A is deflected through an angle of 90°.
The coefficient of restitution between the spheres is e.



(i) Show that $\tan\alpha = \sqrt{\dfrac{2e-1}{3}}$.

(ii) Find e, if the magnitude of the impulse exerted by A on B is $mu\cos\alpha$.

(i) PCM $\quad mu\cos\alpha \;+\; 2m(0) \;=\; mv_1 + 2mv_2$ 5

NEL $\quad v_1 - v_2 = -e(u\cos\alpha - 0)$ 5

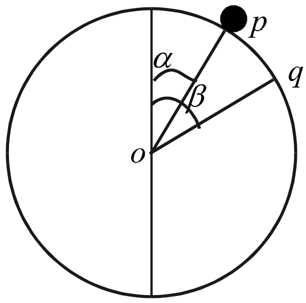
$$v\cos\alpha = u\sin\alpha \quad \textit{and}$$
$$v_1 = -v\sin\alpha$$
$$\Rightarrow \quad v_1 = -u\tan\alpha\sin\alpha$$
5

$$u\cos\alpha = v_1 + 2v_2$$
$$-2eu\cos\alpha = 2v_1 - 2v_2$$
$$u\cos\alpha - 2eu\cos\alpha = 3(-u\tan\alpha\sin\alpha)$$
$$\tan^2\alpha = \frac{2e-1}{3}$$
$$\tan\alpha = \sqrt{\frac{2e-1}{3}}$$
5

(ii) Impulse exerted by A on B $= |2m(0) - 2mv_2|$
$$mu\cos\alpha = 2m(eu\cos\alpha - u\tan\alpha\sin\alpha)$$
$$2\tan^2\alpha = 2e-1$$
$$2(2e-1) = 3(2e-1)$$
$$\Rightarrow \quad e = \tfrac{1}{2}$$
5

25

6. **(b)** A particle of mass m is held at a point p on the surface of a fixed smooth sphere, centre o and radius r.
op makes an angle α with the upward vertical.
The particle is released from rest.
When the particle reaches an arbitrary point q, its speed is v.
oq makes an angle β with the upward vertical.



(i) Show that $v^2 = 2gr(\cos\alpha - \cos\beta)$.

(ii) If $\cos\alpha = \frac{2}{3}$ and if q is the point at which the particle leaves the surface, find the value of β.

(i)

Gain in KE $=$ Loss in PE		
$\frac{1}{2}mv^2 = mg\{r\cos\alpha - r\cos\beta\}$		5, 5
$v^2 = 2gr\{\cos\alpha - \cos\beta\}$		5